MANE 4240 & CIVL 4240
Introduction to Finite Elements

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Four-noded rectangular element

Reading assignment:

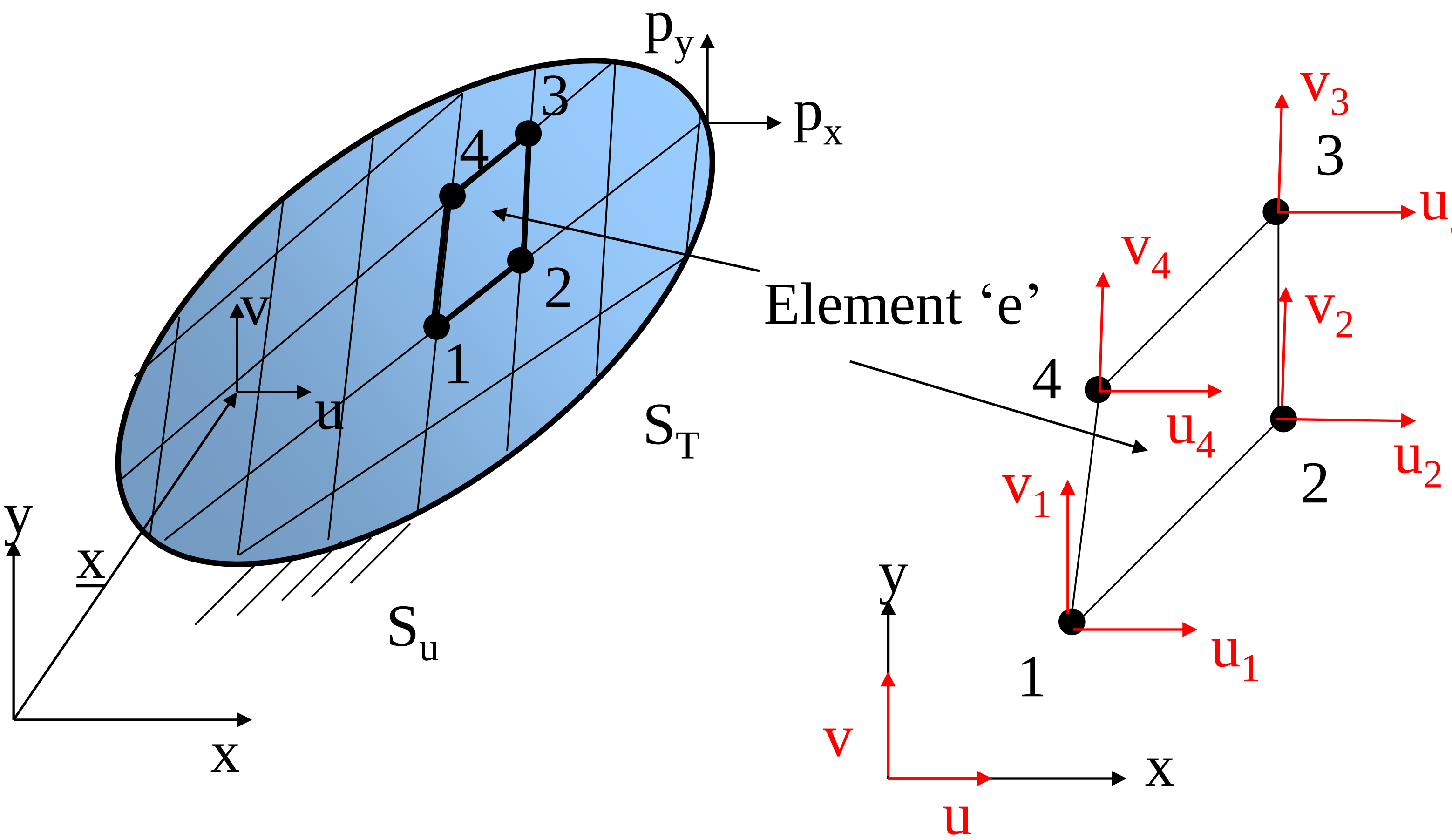
Logan 10.2 + Lecture notes

Summary:

- Computation of shape functions for 4-noded quad
- Special case: rectangular element
- Properties of shape functions
- Computation of strain-displacement matrix
- Example problem
- Hint at how to generate shape functions of higher order (Lagrange) elements

Finite element formulation for 2D:

Step 1: Divide the body into **finite elements** connected to each other through special points ("**nodes**")



$$\underline{d} = \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \\ u_4 \\ v_4 \end{Bmatrix}$$

Summary: For each element

Displacement approximation in terms of shape functions

$$\underline{u} = \underline{N}\ \underline{d}$$

Strain approximation in terms of strain-displacement matrix

$$\underline{\varepsilon} = \underline{B}\ \underline{d}$$

Stress approximation

$$\underline{\sigma} = \underline{D}\underline{B}\ \underline{d}$$

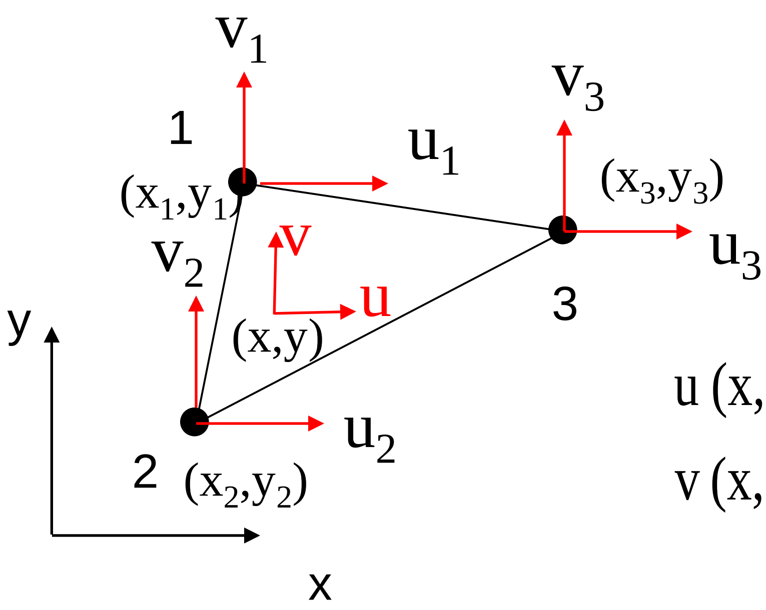
Element stiffness matrix

$$\underline{k} = \int_{V^e} \underline{B}^T \underline{D}\ \underline{B}\ dV$$

Element nodal load vector

$$\underline{f} = \underbrace{\int_{V^e} \underline{N}^T \underline{X}\ dV}_{\underline{f}_b} + \underbrace{\int_{S_T^e} \underline{N}^T \underline{T}_S\ dS}_{\underline{f}_S}$$

Constant Strain Triangle (CST) : Simplest 2D finite element

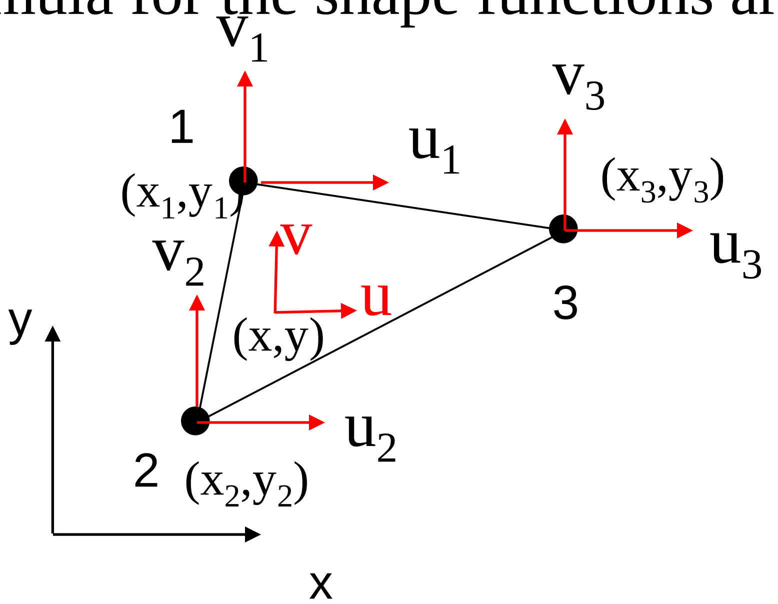


$$u(x,y) \approx N_1(x,y)\,u_1 + N_2(x,y)\,u_2 + N_3(x,y)\,u_3$$

$$v(x,y) \approx N_1(x,y)\,v_1 + N_2(x,y)\,v_2 + N_3(x,y)\,v_3$$

- 3 nodes per element
- 2 dofs per node (each node can move in x- and y- directions)
- Hence 6 dofs per element

Formula for the shape functions are



$$N_1 = \frac{a_1 + b_1 x + c_1 y}{2A}$$

$$N_2 = \frac{a_2 + b_2 x + c_2 y}{2A}$$

$$N_3 = \frac{a_3 + b_3 x + c_3 y}{2A}$$

where

$$A = area\, of\ triangle = \frac{1}{2}\det\begin{bmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{bmatrix}$$

$$a_1 = x_2 y_3 - x_3 y_2 \quad b_1 = y_2 - y_3 \quad c_1 = x_3 - x_2$$

$$a_2 = x_3 y_1 - x_1 y_3 \quad b_2 = y_3 - y_1 \quad c_2 = x_1 - x_3$$

$$a_3 = x_1 y_2 - x_2 y_1 \quad b_3 = y_1 - y_2 \quad c_3 = x_2 - x_1$$

Approximation of **displacements**

$$\underline{u} = \underline{N}\ \underline{d}$$

$$\underline{u} = \begin{Bmatrix} u\,(x,y) \\ v\,(x,y) \end{Bmatrix} = \begin{bmatrix} N_1 & 0 & N_2 & 0 & N_3 & 0 \\ 0 & N_1 & 0 & N_2 & 0 & N_3 \end{bmatrix} \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \end{Bmatrix}$$

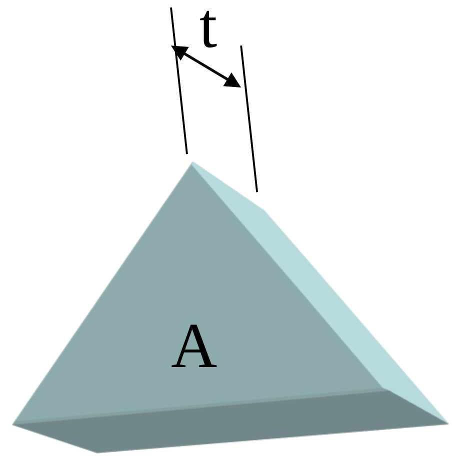
Approximation of the **strains**

$$\underline{\varepsilon} = \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} = \begin{Bmatrix} \dfrac{\partial u}{\partial x} \\ \dfrac{\partial v}{\partial y} \\ \dfrac{\partial u}{\partial y} + \dfrac{\partial v}{\partial x} \end{Bmatrix} \approx \underline{B}\underline{d}$$

$$\underline{B} = \begin{bmatrix} \dfrac{\partial N_1(x,y)}{\partial x} & 0 & \dfrac{\partial N_2(x,y)}{\partial x} & 0 & \dfrac{\partial N_3(x,y)}{\partial x} & 0 \\ 0 & \dfrac{\partial N_1(x,y)}{\partial y} & 0 & \dfrac{\partial N_2(x,y)}{\partial y} & 0 & \dfrac{\partial N_3(x,y)}{\partial y} \\ \dfrac{\partial N_1(x,y)}{\partial y} & \dfrac{\partial N_1(x,y)}{\partial x} & \dfrac{\partial N_2(x,y)}{\partial y} & \dfrac{\partial N_2(x,y)}{\partial x} & \dfrac{\partial N_3(x,y)}{\partial y} & \dfrac{\partial N_3(x,y)}{\partial x} \end{bmatrix} = \frac{1}{2A} \begin{bmatrix} b_1 & 0 & b_2 & 0 & b_3 & 0 \\ 0 & c_1 & 0 & c_2 & 0 & c_3 \\ c_1 & b_1 & c_2 & b_2 & c_3 & b_3 \end{bmatrix}$$

Element stiffness matrix

$$\underline{k} = \int_{V^e} \underline{B}^T \underline{D}\, \underline{B}\, dV$$



Since $\underline{B}$ is constant

$$\underline{k} = \underline{B}^T \underline{D}\, \underline{B} \int_{V^e} dV = \underline{B}^T \underline{D}\, \underline{B}\, At$$

t=thickness of the element
A=surface area of the element

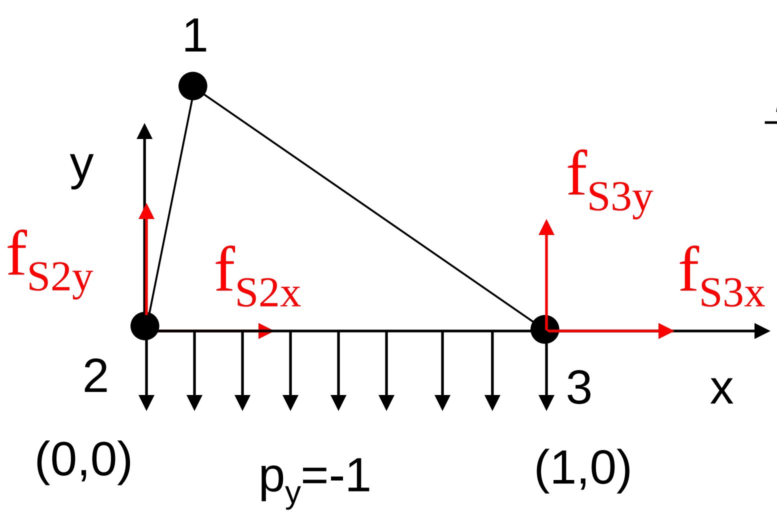
Element nodal load vector

$$\underline{f} = \underbrace{\int_{V^e} \underline{N}^T \underline{X}\, dV}_{\underline{f}_b} + \underbrace{\int_{S_T^e} \underline{N}^T \underline{T}_S\, dS}_{\underline{f}_S}$$

Class exercise

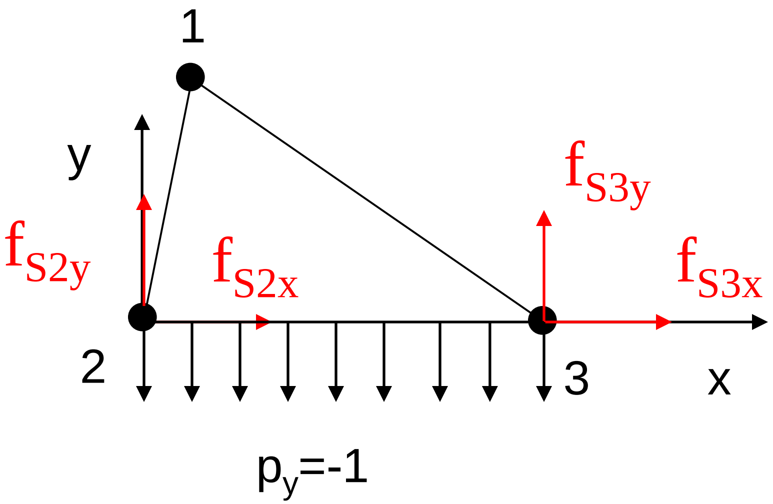
For the CST shown below, compute the vector of nodal loads due to surface traction

$$\underline{f}_S = \int_{S_T^e} \underline{\mathrm{N}}^T \underline{T}_S \, dS$$



$$\underline{f}_S = t\int_{l_{1-3}^e} \underline{\mathrm{N}}^T \Big|_{along\ 2-3} \underline{T}_S \, dS$$

Class exercise



$$\underline{f}_S = t\int_{l_{1-3}^e} \underline{\mathbf{N}}^T\Big|_{along\ 2-3} \underline{T}_S\ dS$$

$$\underline{T}_S = \begin{Bmatrix} 0 \\ -1 \end{Bmatrix}$$

The only nonzero nodal loads are

$$f_{S_{2y}} = t\int_{x=0}^{1} N_2\Big|_{along\ 2-3} p_y\ dx$$

$$f_{S_{3y}} = t\int_{x=0}^{1} N_3\Big|_{along\ 2-3} p_y\ dx$$

$$N_2\Big|_{along\ 2-3} = \left[\frac{a_2+b_2x+c_2y}{2A}\right]_{y=0} = \frac{a_2+b_2x}{2A} = \frac{(x_3y_1-x_1y_3)+(y_3-y_1)x}{2A}$$

$$= \frac{y_1-y_1x}{\det\begin{bmatrix} 1 & \mathrm{x}_1 & y_1 \\ 1 & \mathrm{x}_2 & y_2 \\ 1 & \mathrm{x}_3 & y_3 \end{bmatrix}} = \frac{y_1(1-x)}{\det\begin{bmatrix} 1 & \mathrm{x}_1 & y_1 \\ 1 & \mathrm{x}_2 & 0 \\ 1 & \mathrm{x}_3 & 0 \end{bmatrix}} = \frac{y_1(1-x)}{y_1(x_3-x_2)}$$

$= 1-x$ (can you derive this simpler?)

$$\Rightarrow f_{S_{2y}} = t\int_{x=0}^{1} N_2\Big|_{along\ 2-3} p_y\ dx$$

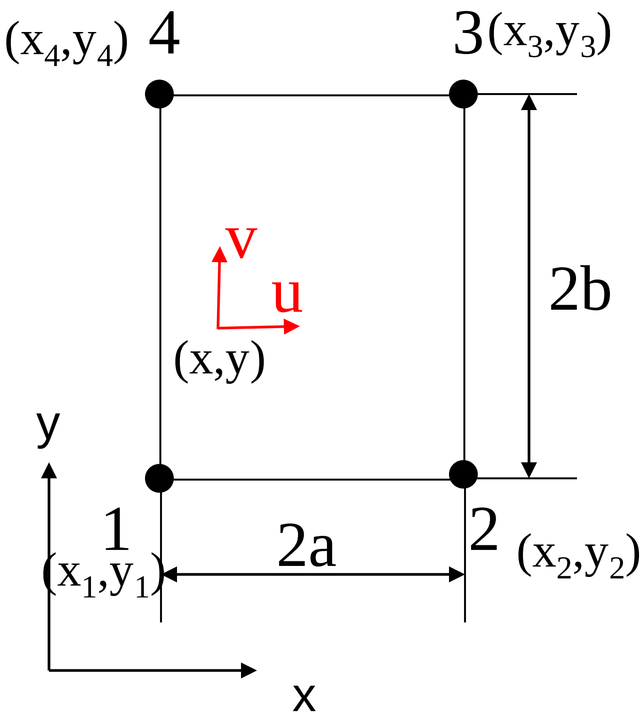
$$= t\int_{x=0}^{1} (1-x)(-1)\ dx$$

$$= -\frac{t}{2}$$

Now compute

$$f_{S_{3y}} = t\int_{x=0}^{1} N_3\Big|_{along\ 2-3} p_y\ dx$$

4-noded rectangular element with edges parallel to the coordinate axes:



$$u\,(x,y) \approx \sum_{i=1}^{4} N_i(x,y)u_i$$

$$v\,(x,y) \approx \sum_{i=1}^{4} N_i(x,y)v_i$$

- 4 nodes per element
- 2 dofs per node (each node can move in x- and y- directions)
- 8 dofs per element

Using similar arguments, choose

$$N_1 = \frac{1}{4ab}(x - x_2)(y - y_4)$$

$$N_2 = -\frac{1}{4ab}(x - x_1)(y - y_3)$$

$$N_3 = \frac{1}{4ab}(x - x_4)(y - y_2)$$

$$N_4 = -\frac{1}{4ab}(x - x_3)(y - y_1)$$

Properties of the shape functions:

1. The shape functions N_1, N_2 , N_3 and N_4 are bilinear functions of x and y

2. Kronecker delta property

$$N_i(x,y) = \begin{cases} 1 & at\ node\ 'i' \\ 0 & at\ other\ nodes \end{cases}$$

3. Completeness

$$\sum_{i=1}^{4} N_i = 1$$

$$\sum_{i=1}^{4} N_i x_i = x$$

$$\sum_{i=1}^{4} N_i y_i = y$$

3. Along lines parallel to the x- or y-axes, the shape functions are linear. But along any other line they are nonlinear.

4. An element shape function related to a specific nodal point is zero along element boundaries not containing the nodal point.

5. The displacement field is continuous across elements

6. The strains and stresses are not constant within an element nor are they continuous across element boundaries.

The strain-displacement relationship

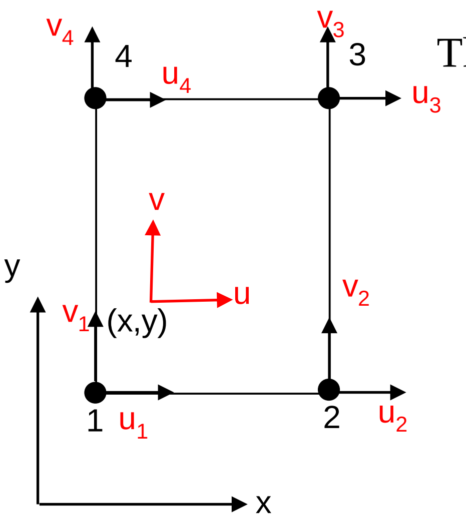
$$\underline{\varepsilon} = \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix}$$

$$= \underbrace{\left[\begin{array}{cc:cc:cc:cc} \frac{\partial N_1(x,y)}{\partial x} & 0 & \frac{\partial N_2(x,y)}{\partial x} & 0 & \frac{\partial N_3(x,y)}{\partial x} & 0 & \frac{\partial N_4(x,y)}{\partial x} & 0 \\ 0 & \frac{\partial N_1(x,y)}{\partial y} & 0 & \frac{\partial N_2(x,y)}{\partial y} & 0 & \frac{\partial N_3(x,y)}{\partial y} & 0 & \frac{\partial N_4(x,y)}{\partial y} \\ \frac{\partial N_1(x,y)}{\partial y} & \frac{\partial N_1(x,y)}{\partial x} & \frac{\partial N_2(x,y)}{\partial y} & \frac{\partial N_2(x,y)}{\partial x} & \frac{\partial N_3(x,y)}{\partial y} & \frac{\partial N_3(x,y)}{\partial x} & \frac{\partial N_4(x,y)}{\partial y} & \frac{\partial N_4(x,y)}{\partial x} \end{array}\right]}_{\underline{B}} \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \\ u_4 \\ v_4 \end{Bmatrix}$$

$$\underline{B} = \frac{1}{4ab}\begin{bmatrix} y-y_4 & 0 & y_3-y & 0 & y-y_2 & 0 & y_1-y & 0 \\ 0 & x-x_2 & 0 & x_1-x & 0 & x-x_4 & 0 & x_3-x \\ x-x_2 & y-y_4 & x_1-x & y_3-y & x-x_4 & y-y_2 & x_3-x & y_1-y \end{bmatrix}$$

Notice that the strains (and hence the stresses) are NOT constant within an element

Computation of the terms in the stiffness matrix of 2D elements (recap)



The B-matrix (strain-displacement) corresponding to this element is

u_1	v_1	u_2	v_2	u_3	v_3	u_4	v_4
$\frac{\partial N_1(x,y)}{\partial x}$	0	$\frac{\partial N_2(x,y)}{\partial x}$	0	$\frac{\partial N_3(x,y)}{\partial x}$	0	$\frac{\partial N_4(x,y)}{\partial x}$	0
0	$\frac{\partial N_1(x,y)}{\partial y}$	0	$\frac{\partial N_2(x,y)}{\partial y}$	0	$\frac{\partial N_3(x,y)}{\partial y}$	0	$\frac{\partial N_4(x,y)}{\partial y}$
$\frac{\partial N_1(x,y)}{\partial y}$	$\frac{\partial N_1(x,y)}{\partial x}$	$\frac{\partial N_2(x,y)}{\partial y}$	$\frac{\partial N_2(x,y)}{\partial x}$	$\frac{\partial N_3(x,y)}{\partial y}$	$\frac{\partial N_3(x,y)}{\partial x}$	$\frac{\partial N_4(x,y)}{\partial y}$	$\frac{\partial N_4(x,y)}{\partial x}$

We will denote the columns of the B-matrix as

$$\underline{B}_{u_1} = \begin{bmatrix} \frac{\partial N_1(x,y)}{\partial x} \\ 0 \\ \frac{\partial N_1(x,y)}{\partial y} \end{bmatrix}; \ \underline{B}_{v_1} = \begin{bmatrix} 0 \\ \frac{\partial N_1(x,y)}{\partial y} \\ \frac{\partial N_1(x,y)}{\partial x} \end{bmatrix}; \text{ and so on...}$$

The **stiffness matrix** corresponding to this element is

$$\boxed{\underline{k} = \int_{V^e} \underline{B}^T \underline{D}\ \underline{B}\, dV}$$

which has the following form

$$
\begin{array}{c}
\quad\ u_1 \qquad v_1 \qquad u_2 \qquad v_2 \qquad u_3 \qquad v_3 \qquad u_4 \qquad v_4 \\
\underline{k} = \begin{bmatrix}
k_{11} & k_{12} & k_{13} & k_{14} & k_{15} & k_{16} & k_{17} & k_{18} \\
k_{21} & k_{22} & k_{23} & k_{24} & k_{25} & k_{26} & k_{27} & k_{28} \\
k_{31} & k_{32} & k_{33} & k_{34} & k_{35} & k_{36} & k_{37} & k_{38} \\
k_{41} & k_{42} & k_{43} & k_{44} & k_{45} & k_{46} & k_{47} & k_{48} \\
k_{51} & k_{52} & k_{53} & k_{54} & k_{55} & k_{56} & k_{57} & k_{58} \\
k_{61} & k_{62} & k_{63} & k_{64} & k_{65} & k_{66} & k_{67} & k_{68} \\
k_{71} & k_{72} & k_{73} & k_{74} & k_{75} & k_{76} & k_{77} & k_{78} \\
k_{81} & k_{82} & k_{83} & k_{84} & k_{85} & k_{86} & k_{87} & k_{88}
\end{bmatrix}
\begin{array}{c} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \\ u_4 \\ v_4 \end{array}
\end{array}
$$

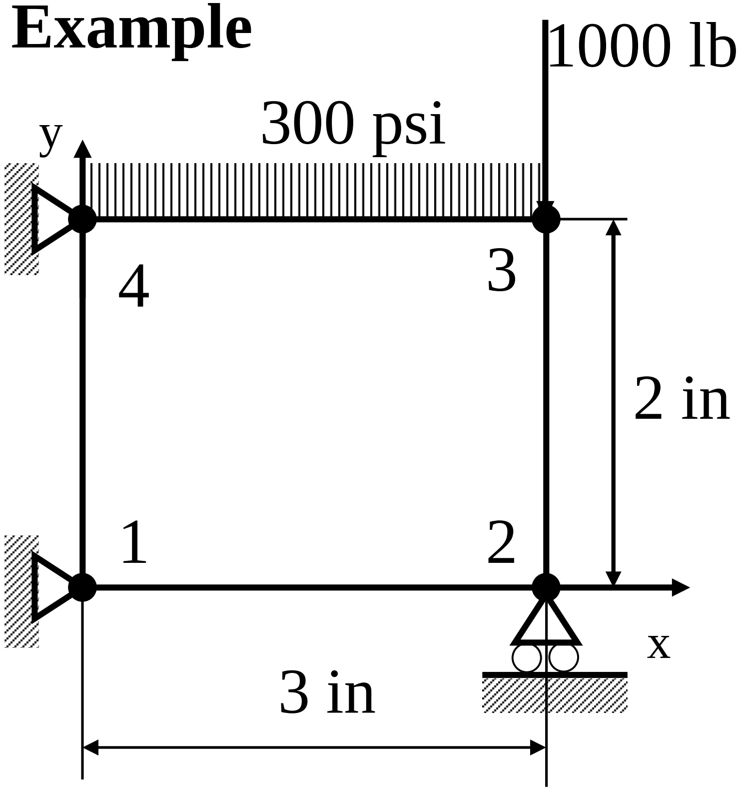
The individual entries of the stiffness matrix may be computed as follows

$$k_{11} = \int_{V^e} \underline{B}_{u_1}{}^T \underline{D}\ \underline{B}_{u_1}\ dV; \quad k_{12} = \int_{V^e} \underline{B}_{u_1}{}^T \underline{D}\ \underline{B}_{v_1}\ dV; k_{13} = \int_{V^e} \underline{B}_{u_1}{}^T \underline{D}\ \underline{B}_{u_2}\ dV, ...$$

$$k_{21} = \int_{V^e} \underline{B}_{v_1}{}^T \underline{D}\ \underline{B}_{u_1}\ dV; k_{21} = \int_{V^e} \underline{B}_{v_1}{}^T \underline{D}\ \underline{B}_{v_1}\ dV; .....$$

Notice that these formulae are quite general (apply to all kinds of finite elements, CST, quadrilateral, etc) since we have not used any specific shape functions for their derivation.

Example



Thickness (t) = 0.5 in
E= 30×10^6 psi
ν=0.25

(a) Compute the unknown nodal displacements.
(b) Compute the stresses in the two elements.

This is exactly the same problem that we solved in last class, except now we have to use a **single** 4-noded element

Realize that this is a plane stress problem and therefore we need to use

$$\underline{D}=\frac{E}{1-\nu^2}\begin{bmatrix}1 & \nu & 0\\ \nu & 1 & 0\\ 0 & 0 & \dfrac{1-\nu}{2}\end{bmatrix}=\begin{bmatrix}3.2 & 0.8 & 0\\ 0.8 & 3.2 & 0\\ 0 & 0 & 1.2\end{bmatrix}\times 10^7\ psi$$

Write down the shape functions

$$N_1=\frac{1}{4ab}(x-x_2)(y-y_4)=\frac{(x-3)(y-2)}{6}$$

$$N_2=-\frac{1}{4ab}(x-x_1)(y-y_3)=-\frac{x(y-2)}{6}$$

$$N_3=\frac{1}{4ab}(x-x_4)(y-y_2)=\frac{xy}{6}$$

$$N_4=-\frac{1}{4ab}(x-x_3)(y-y_1)=-\frac{(x-3)y}{6}$$

x	y
0	0
3	0
3	2
0	2

We have 4 nodes with 2 dofs per node=8dofs. However, 5 of these are fixed. The nonzero displacements are

u_2 u_3 v_3

Hence we need to solve

$$\begin{array}{c|c|c} u_2 & u_3 & v_3 \end{array}$$

$$\begin{bmatrix} k_{11} & k_{12} & k_{13} \\ k_{21} & k_{22} & k_{23} \\ k_{31} & k_{32} & k_{33} \end{bmatrix} \begin{Bmatrix} u_2 \\ u_3 \\ v_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ f_{3y} \end{Bmatrix}$$

Need to compute only the **relevant terms in the stiffness matrix**

$$k_{11} = \int_{V^e} \underline{\mathrm{B}}_{u_2}{}^{\mathrm{T}} \underline{\mathrm{D}} \; \underline{\mathrm{B}}_{u_2} \; \mathrm{dV}; \quad k_{12} = \int_{V^e} \underline{\mathrm{B}}_{u_2}{}^{\mathrm{T}} \underline{\mathrm{D}} \; \underline{\mathrm{B}}_{u_3} \; \mathrm{dV}; \; k_{13} = \int_{V^e} \underline{\mathrm{B}}_{u_2}{}^{\mathrm{T}} \underline{\mathrm{D}} \; \underline{\mathrm{B}}_{v_3} \; \mathrm{dV}$$

$$k_{21} = \int_{V^e} \underline{\mathrm{B}}_{u_3}{}^{\mathrm{T}} \underline{\mathrm{D}} \; \underline{\mathrm{B}}_{u_2} \; \mathrm{dV}; \quad k_{22} = \int_{V^e} \underline{\mathrm{B}}_{u_3}{}^{\mathrm{T}} \underline{\mathrm{D}} \; \underline{\mathrm{B}}_{u_3} \; \mathrm{dV}; \; k_{13} = \int_{V^e} \underline{\mathrm{B}}_{u_3}{}^{\mathrm{T}} \underline{\mathrm{D}} \; \underline{\mathrm{B}}_{v_3} \; \mathrm{dV}$$

$$k_{31} = \int_{V^e} \underline{\mathrm{B}}_{v_3}{}^{\mathrm{T}} \underline{\mathrm{D}} \; \underline{\mathrm{B}}_{u_2} \; \mathrm{dV}; \quad k_{22} = \int_{V^e} \underline{\mathrm{B}}_{v_3}{}^{\mathrm{T}} \underline{\mathrm{D}} \; \underline{\mathrm{B}}_{u_3} \; \mathrm{dV}; \; k_{13} = \int_{V^e} \underline{\mathrm{B}}_{v_3}{}^{\mathrm{T}} \underline{\mathrm{D}} \; \underline{\mathrm{B}}_{v_3} \; \mathrm{dV}$$

Compute only the relevant columns of the B matrix

$$\underline{B}_{u_2} = \begin{Bmatrix} \dfrac{\partial N_2}{\partial x} \\ 0 \\ \dfrac{\partial N_2}{\partial y} \end{Bmatrix} = \begin{Bmatrix} \dfrac{(2-y)}{6} \\ 0 \\ -\dfrac{x}{6} \end{Bmatrix}$$

$$\underline{B}_{u_3} = \begin{Bmatrix} \dfrac{\partial N_3}{\partial x} \\ 0 \\ \dfrac{\partial N_3}{\partial y} \end{Bmatrix} = \begin{Bmatrix} \dfrac{y}{6} \\ 0 \\ \dfrac{x}{6} \end{Bmatrix}$$

$$\underline{B}_{v_3} = \begin{Bmatrix} 0 \\ \dfrac{\partial N_3}{\partial y} \\ \dfrac{\partial N_3}{\partial x} \end{Bmatrix} = \begin{Bmatrix} 0 \\ \dfrac{x}{6} \\ \dfrac{y}{6} \end{Bmatrix}$$

$$k_{11} = \int_{V^e} \underline{\mathrm{B}}_{u_2}{}^{\mathrm{T}} \underline{\mathrm{D}} \; \underline{\mathrm{B}}_{u_2} \; \mathrm{dV}$$

$$= 0.5 \int_{x=0}^{3} \int_{y=0}^{2} \left[(0.1067\times10^8 - 0.533\times10^7)(\frac{2-y}{6}) + 3.33\times10^5 x^2 \right] dxdy$$

$$= 0.656\times10^7$$

Similarly compute the other terms

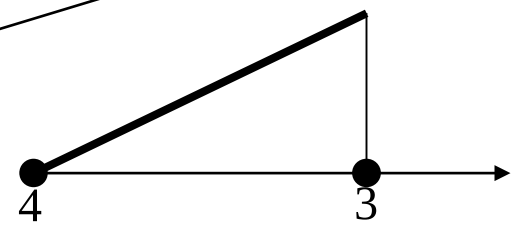
How do we compute f_{3y}

$$f_{3y} = -1000 + f_{S_{3y}}$$

$$N_3\big|_{\substack{edge\\4-3}} = N_3\big|_{y=2} = \left[\frac{xy}{6}\right]_{y=2} = \frac{x}{3}$$

$$f_{S_{3y}} = t\int_{x=0}^{3} N_3\big|_{\substack{along\\edge\,3-4}} (-300)\,dx$$

$$= (0.5)(-300)\int_{x=0}^{3} \frac{x}{3}\,dx$$



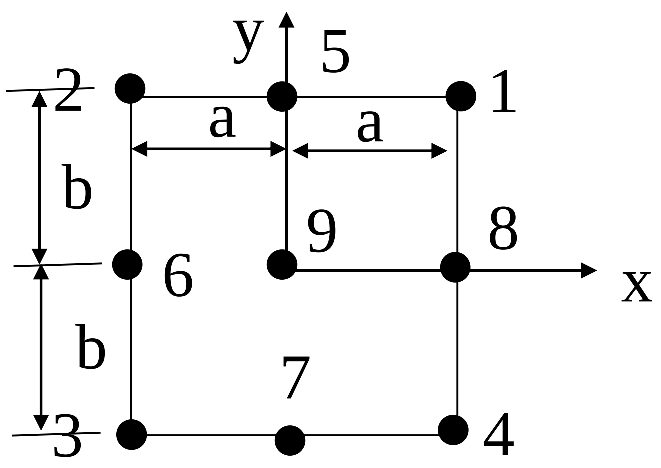
$$= -150\times\frac{3}{2}$$

$$= -225\ \ lb$$

$$\Rightarrow\ f_{3y} = -1000 + f_{S_{3y}} = -1225\ lb$$

How about a **9-noded rectangle**?

Corner nodes



$$N_1 = \left[\frac{x(a+x)}{2a^2}\right]\left[\frac{y(b+y)}{2b^2}\right] \quad N_2 = \left[-\frac{x(a-x)}{2a^2}\right]\left[\frac{y(b+y)}{2b^2}\right]$$

$$N_3 = \left[-\frac{x(a-x)}{2a^2}\right]\left[-\frac{y(b-y)}{2b^2}\right] \quad N_4 = \left[\frac{x(a+x)}{2a^2}\right]\left[-\frac{y(b-y)}{2b^2}\right]$$

Midside nodes

$$N_5 = \left[\frac{a^2-x^2}{a^2}\right]\left[\frac{y(b+y)}{2b^2}\right] \quad N_6 = \left[-\frac{x(a-x)}{2a^2}\right]\left[\frac{b^2-y^2}{b^2}\right]$$

$$N_7 = \left[\frac{a^2-x^2}{a^2}\right]\left[-\frac{y(b-y)}{2b^2}\right] \quad N_8 = \left[\frac{x(a+x)}{2a^2}\right]\left[\frac{b^2-y^2}{b^2}\right]$$

Center node

$$N_9 = \left[\frac{a^2-x^2}{a^2}\right]\left[\frac{b^2-y^2}{b^2}\right]$$

Question: Can you generate the shape functions of a 16-noded rectangle?

Note: These elements, whose shape functions are generated by multiplying the shape functions of 1D elements, are said to belong to the **"Lagrange" family**